1 Exercises 1 - problems

1.1 Nonlinear extension of the Lorentz model

In the lecture, we discussed a classcial model for nonlinear optical effects based on nonlinear modifications of the Lorenz model. In this context, we found several solutions for nonlinear effects like e.g. SFG. These solutions can be associated with susceptibilities of the medium.

In particular, we said that:

$$P^{(k)}(\omega) = -Nex^{(k)}(\omega). \tag{1.1}$$

For example, this rather general relation is a representation of the following more specific cases:

$$P^{(1)}(\omega_j) = -Nex^{(1)}(\omega_j) \tag{1.2a}$$

$$P^{(2)}(\omega_1 + \omega_2) = -Nex^{(2)}(\omega_1 + \omega_2). \tag{1.2b}$$

Use the corresponding equations like 1.2a and 1.2b in combination with adequate equations relating the polarization $P^{(k)}$ of the medium with the electric field like, for example:

$$P^{(1)}(\omega_j) = \epsilon_0 \chi^{(1)}(\omega_j) E(\omega_j) \tag{1.3a}$$

$$P^{(2)}(\omega_1 + \omega_2) = \epsilon_0 \chi^{(2)}(\omega_1 + \omega_2, \omega_1, \omega_2) E(\omega_1) E(\omega_2). \tag{1.3b}$$

Calculate: the expressions for $\chi^1(\omega_j)$, $\chi^2(\omega_1+\omega_2, \omega_1, \omega_2)$, $\chi^2(\omega_1-\omega_2, \omega_1, \omega_2)$, $\chi^2(0, \omega_j, -\omega_j)$.

1.2 The susceptibilities and the complex denominator functions

This is related to our discussion of an anharmonic oscillator model for nonlinear optical effects in noncentrosymmetric media. **Compare** the expressions

$$P^{(1)}(\omega_j) = -Nex^{(1)}(\omega_j) \tag{1.4}$$

$$P^{(1)}(\omega_j) = \epsilon_0 \chi^{(1)}(\omega_j) E(\omega_j) \tag{1.5}$$

as well as

$$P^{(2)}(2\omega_j) = -Nex^{(2)}(2\omega_j) \tag{1.6}$$
$$P^{(2)}(2\omega_j) = \epsilon_0\chi^{(2)}(2\omega_j, \omega_j\omega_j)E(\omega_j)^2, \tag{1.7}$$

to find expressions for the susceptibilities $\chi^{(1)}(\omega_j)$ and $\chi^{(2)}(2\omega_j, \omega_j\omega_j)$.

1.3 Miller's rule

There is an empirical rule going back to Miller[1]. Miller noted that the quantity

$$\frac{\chi^{(2)}(\omega_1+\omega_2, \omega_1, \omega_2)}{\chi^{(1)}(\omega_1+\omega_2)\chi^{(1)}(\omega_1)\chi^{(1)}(\omega_2)} \tag{1.8}$$

is approximately constant for all noncentrosymmetric crystals. If we compare that with the relation

$$x^{(2)}(\omega_1+\omega_2) = -\frac{2a(e/m)^2E_1E_2}{D(\omega_1+\omega_2)D(\omega_1)D(\omega_2)} \tag{1.9}$$

we saw during the lecture, and if we take into account the relation between the various Ds and the corresponding susceptibilities, which we derived in Problem 2 above, then we can conclude something about a combination of material and fundamental constants m, a, ϵ_0, N, and e. Give that expression and discuss what we can conclude about it.

We can even go one step further: if we assume that the linear and the nonlinear contributions to the restoring force given by:

$$F = -m\omega_0^2x - max^2 \tag{1.10}$$

become approximately equal if the displacement x becomes comparable to the distance between atoms d in the medium, then we can make an order-of-magnitude estimate of the nonlinear coefficient a.

2 Exercises 2 - problems

2.1 The electro-optic modulator and the indicatrix

Consider the equation we had for the indicatrix in the principal-axis coordinate system for KDP:

$$\frac{X^2}{n_o^2} + \frac{Y^2}{n_o^2} + \frac{Z^2}{n_e^2} + 2r_{63}E_Z XY = 1. \tag{2.1}$$

If we do a rotation into a new coordinate system x, y, z, we can ensure that the indicatrix becomes diagonal again. That means, such that equation 2.1 becomes:

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_e^2} = 1, \tag{2.2}$$

where the refractive indices along the three coordinate axes are given by $n_z = n_e$ and:

$$\begin{aligned} n_x &= n_o - \frac{1}{2} n_o^3 r_{63} E_z \\ n_y &= n_o + \frac{1}{2} n_o^3 r_{63} E_z. \end{aligned} \tag{2.3}$$

Find out the relation between x, y, z and X, Y, Z. Show that the expressions for n_x,n_y and n_z are correct.

2.2 Intensity modulation using an EOM - I/II

Assume I give you an EOM consisting of a KDP crystal where the beam travels along the Z axis of the crystal and you can apply arbitrary voltages along that direction.

Part I: How could you use that to build an intensity modulator? That means, such that the voltage you apply will directly modulates the intensity of light passing through the crystal.

Part II: Assume that the wavelength of your light is 532 nm. Calculate the half-wave voltage:

$$V_{\lambda/2} = \frac{\pi c}{\omega n_o^3 r_{63}}. \tag{2.4}$$

2.3 Intensity modulation using an EOM - II/II

Assume we use a Pockels cell to build an intensity modulator. As we vary the applied voltave from 0 V to $V_{\lambda/2}$, how will the transmitted power change?

If you wanted to use the same modulator as in the prior exercise such that a (not too large) change in the voltage results in a change in the transmitted power that is directly proportional to the change of voltage, how would you do that?

2.4 Bragg scattering off an acoustic wave

Examples for representative values one can encounter in a laboratory are[2] $v = 1.5 \times 10^3$ m/s, $\Omega/2\pi = 200$ MHz. **(1)** Calculate the acoustic wavelength Λ. **(2)** Assuming that the optical wavelength is 532 nm, calculate the Bragg diffraction angle θ.

2.5 Stokes scattering

Assume we have stokes scattering at an angle θ such that $\vec{k}' = \vec{k} - \vec{q}$, Write $|q|$ as a function of $|k|$ and θ. Using the dispersion relation $\Omega = |\vec{q}|v$, write Ω as a function of θ. What can we conclude about the maximum frequency shift $\Omega_{\max}$ we can get? Describe the different cases of froward scattering and backward scattering. Assuming that $\lambda = 1$ m, $v = 10^3$ m/s and $n = 1.5$, calculate $\Omega_{\max}$.

2.6 Conversion efficiency in an AOM

Taking the equation

$$|\kappa| = \frac{\omega\gamma_e}{2nc\cos\theta}\left(\frac{I}{2Kv}\right)^{1/2}. \tag{2.5}$$

calculate $|\kappa|$ using the following parameters: $n = 1.33$, $\gamma_e = 0.82$, $v = 1.5 \times 10^3$ m/s, and $K = 2.19 \times 10^1 1\,\mathrm{m}^2/\mathrm{N}$. Assume that the optical wavelength is 500 nm, that $\cos\theta \approx 1$, and that the acoustic intensity is $I \approx 1\,\mathrm{W/cm}^2$. Given these values, what is the value of $|\kappa|$, and what optimum conversion efficiency can we achieve? For the latter, use the following equation:

$$\eta = \sin^2\left(|\kappa|L\right). \tag{2.6}$$

3 Exercises 3 - problems

3.1 The coupled wave equation for SFG

Take the wave equation:

$$\vec{\nabla}^2 \vec{E}_n(\vec{r}) + \frac{\omega_n^2}{c^2} \vec{\epsilon}^{(1)}(\omega_n) \cdot \vec{E}_n(\vec{r}) = -\frac{\omega_n^2}{\epsilon_0 c^2} \vec{P}_n^{\mathrm{NL}}(\vec{r}) \tag{3.1}$$

adapt it according to the following considerations. The wave equation must hold for all frequencies involved. We assume that the field polarization etc are well defined and we can look only at a scalar electric field of the form

$$\begin{aligned} E_i(z,t) &= E_i(z) e^{-i\omega_i t} + \text{c.c.}, \\ E_i(z) &= A_i(z) e^{ik_i t} + \text{c.c.}. \end{aligned} \tag{3.2}$$

The dispersion relation is:

$$k = \frac{n\omega}{c},\ n^2 = \epsilon^{(1)}(\omega). \tag{3.3}$$

We can write the polarization of the SFG field at $\omega_3 = \omega_1 + \omega_2$ as:

$$P_3(z,t) = P_3(z) e^{-i\omega_3 t} + \text{c.c.} = 4\epsilon_0 d_{\text{eff}} E_1(z) E_2(z).$$

Show that we can then write the wave equation as:

$$\frac{d^2 A_3}{dz^2} + 2ik_3 \frac{dA_3}{dz} = -4 \frac{d_{\text{eff}} \omega_3^2}{c^2} A_1 A_2 e^{i(k_1+k_2-k_3)z}, \tag{3.4}$$

3.2 Second harmonic generation (SHG)

Let us assume we want to do SHG, and that $A_2(0) = 0$. That means, at the entry face of the crystal we only have the fundamental field but not yet an SHG field. We can estimate $A_1(0)$ from the power P and the waist size w_0 of the incoming beam:

$$I_1 = \frac{P}{\pi w_0^2} = 2n_1 \epsilon_0 c A_1^2. \tag{3.5}$$

Let us assume that the depth b of the focal region is equal to the crystal length L:

$$b = \frac{2\pi w_0^2}{\lambda_1/n_1} = L. \tag{3.6}$$

Calculate expressions for A_1, and for $\zeta = L/l$, where:

$$l = \frac{(n_1 n_2)^{1/2} c}{2\omega_1 d_{\text{eff}} |A_1(0)|}. \tag{3.7}$$

3.3 SHG efficiency

Using the result for ζ from the previous exercise, calculate the efficiency of SHG for a cw laser beam with wavelength $\lambda = 500\,\text{nm}$, $P = 1\,\text{W}$, $L = 1\,\text{cm}$, and $d_{\text{eff}} = 4 \times 10^{-12}\,\text{m/V}$. For simplicity, assume that the two refractive indices $n_{1,2} = 2$.

Calculate the efficiency given that:

$$u_1(\zeta) = \operatorname{sech}\zeta = \frac{1}{\cosh\zeta}$$
$$u_2(\zeta) = \tanh\zeta,$$

and assuming that the efficiency is:

$$\eta = \frac{u_2^2(L)}{u_1^2(0)}. \tag{3.8}$$

3.4 Difference frequency generation (DFG)

For perfect phase matching ($\Delta k = k_3 - k_1 - k_2 = 0$), combine the equations:

$$\begin{aligned} \frac{dA_1}{dz} &= \frac{2i\omega_1^2 d_{\text{eff}}}{k_1 c^2} A_3 A_2^* e^{i\Delta k z}, \\ \frac{dA_2}{dz} &= \frac{2i\omega_2^2 d_{\text{eff}}}{k_2 c^2} A_3 A_1^* e^{i\Delta k z}, \end{aligned} \tag{3.9}$$

to get:

$$\frac{d^2 A_2}{dz^2} = \kappa^2 A_2. \tag{3.10}$$

Calculate κ^2.

3.5 OPO threshold

During the lecture, we concluded that the theshold condition for an OPO is given by:

$$\cosh gL = 1 + \frac{l_1 l_2}{2 - l_1 - l_2}. \tag{3.11}$$

Let us consider two special cases: the doubly resonant oscillator and the singly resonant oscillator. As the names suggest, the doubly resonant oscillator is resonant for the idler and the signal wavelength. We can accommodate this assumption by requiring $l_1, l_2 \ll 1$. In the singly resonant case, the resonator is only resonant for the signal wavelength - that is $l_1 \ll 1$.

Show what the threshold condition 3.11 looks like in these two cases, and compare the theshold values for the two cases

4 Exercises 4 - problems

4.1 Intensity-dependent refractive index

During the lecture, we saw that the intensity-dependent refractive index can be written as:

$$n = n_0 + 2\bar{n}_2 |E(\omega)|^2. \tag{4.1}$$

The effective susceptibility was given by:

$$\chi_{\text{eff}} = \chi^{(1)} + 3\chi^{(3)} |E(\omega)|^2. \tag{4.2}$$

At the same time, it is generally true that

$$n^2 = 1 + \chi_{\text{eff}}. \tag{4.3}$$

By inserting equations 4.1 and 4.2 into equation 4.3, find expressions for n as a function of $\chi^{(1)}$ as well as an expression for $\bar{n}_2$ as a function of $\chi^{(3)}$ and n_0.

4.2 Self-focusing of light

Let us assume we have a laser beam travelling through air. For the optical properties of air we assume $n_0 = 1.0003$ and $\chi^{(3)} = 1.7 \times 10^{-25}\,\text{m}^2/\text{V}^2$[2]. If we assume a waist size of $w_0 = 1\,\text{mm}$ and a wavelength $\lambda_0 = 500\,\text{nm}$, calculate the critical power:

$$P_{\text{cr}} = \frac{\pi (0.61)^2 \lambda_0^2}{8 n_0 n_2}. \tag{4.4}$$

If we then assume that our laser is pulsed with 100 fs pulses, where each pulse contains 25 mJ if energy, calculate the peak power P of these pulses and use that to calculate the self-focusing distance:

$$z_{\text{sf}} = w_0 \sqrt{\frac{n_0}{2 n_2 I}} = \frac{2 n_0 w_0^2}{\lambda_0} \frac{1}{\sqrt{P/P_{\text{cr}}}}. \tag{4.5}$$

4.3 Self-trapping of light

We saw during the lecture that self-trapping occurs in a nonlinear medium if the laser beam has a power of:

$$P_{\text{cr}} = \frac{\pi}{4} d^2 I = \frac{\pi (0.61)^2 \lambda_0^2}{8 n_0 n_2} \tag{4.6}$$

Given that $n_2 = 3.2 \times 10^{-18}\,\text{m}^2/\text{W}$ (for carbon disulfide) and that $n_0 = 1.7$, what is the critical power to trap light with a wavelength of $1\,\mu$m)?

Consider we have a pulsed laser with a very moderate pulse width of 10 ns. Suppose that the energy per pulse is 10 mJ. Calculate the peak power of such a pulse. Given the critical power and using equation 4.5, calculate the self-focusing distance in carbon disulfide if our beam enters the crystal with a waist size $w_0 = 100$ m.

4.4 Spatial solitons and self-trapping of light

The solition we got for a spatial soliton looked as follows:

$$A(x, z) = A_0 \text{sech}(x/x_0) e^{i\gamma z}, \tag{4.7}$$

where $\text{sech}(x) = (\cosh x)^{-1}$. The width of this peaked function is determined by x_0:

$$x_0 = \frac{1}{k_0} \left(\frac{n_0}{2\bar{n}_2 |A_0|^2} \right)^{1/2}. \tag{4.8}$$

First, express $|A_0|^2$ in terms of the beam intensity and the beam power by using the relation:

$$I = \frac{P}{\pi w_0^2} = 2 n \epsilon_0 |A_0|^2. \tag{4.9}$$

Then use this expression to write x_0^2 in terms of P, and then set $P = P_{\text{cr}}$ as it is defined in equation 4.4. Assuming $n_0 = 1.7$ and $w_0 = 100\,\mu$m, what value will we get for x_0?

Bibliography

[1] R. C. Miller. Optical second harmonic generation in piezoelectric crystals. *Appl. Phys. Lett.*, 5:17, 1964. 2

[2] Robert W. Boyd. *Nonlinear Optics*. Academic Press, Inc., USA, 3rd edition, 2008. 4, 9